Smooth functions associated with wavelet sets on $\mathbb{R}^d$, $d \geq 1$, and frame bound gaps

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Outline

Problem

- $2^d - 1$ wavelets are required to provide a wavelet orthonormal basis (ONB) with an MRA for $L^2(\mathbb{R}^d)$ ([Madych 92], [Auscher 95], and [Strohmer 93]).
- Until the mid-1990s, it was assumed that it would be impossible to construct a single dyadic wavelet ψ generating an ONB for $L^2(\mathbb{R}^d)$.
- Dai and Larson [92 - 98] discovered wavelet sets, which generated single dyadic (non-MRA) wavelet systems. However, the earliest known examples had complicated spectral properties.
- The Neighborhood Mapping Construction [Benedetto and Leon 01] yields frame wavelet sets with nice spectral properties. We wish to smooth these frame wavelets, but unexpected results occur.

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Wavelet and Fourier transform definition

Definition

For $\psi \in L^2\left(\mathbb{R}^d\right)$, let

$$\mathcal{W}(\psi) = \{D_n T_k \psi(x) = 2^{n/2}\psi(2^n x - k) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

If $\mathcal{W}(\psi)$ forms an orthonormal basis for $L^2\left(\mathbb{R}^d\right)$, then ψ is an *orthonormal dyadic wavelet* or simply a *wavelet*.

For $f \in L^1(\mathbb{R}^d)$, we define the *Fourier transform* of f to be

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int f(x) e^{-2\pi i x \cdot \gamma} dx.$$

By Plancherel's Theorem, $\mathcal{F}$ extends from $L^1 \cap L^2$ to a unitary operator $L^2 \to L^2$. We denote the inverse Fourier transform of a function $g \in L^2(\widehat{\mathbb{R}}^d)$ as $\mathcal{F}^{-1} g = \check{g}$.

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Examples of wavelets

Example

- The *Haar wavelet* is $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)} \in L^2(\mathbb{R})$.
- The *Shannon* or *Littlewood-Paley wavelet* is $\check{\mathbb{1}}_{[-1,-1/2)\cup[1/2,1)}$.
- The *Journé wavelet* is $\check{\mathbb{1}}_{[-\frac{16}{7},-2)\cup[-\frac{1}{2},-\frac{2}{7})\cup[\frac{2}{7},\frac{1}{2})\cup[2,\frac{16}{7})}$.

Definition

If $K \subseteq \widehat{\mathbb{R}}^d$ is measurable and $\check{\mathbb{1}}_K$ is a wavelet for $L^2(\mathbb{R}^d)$, then K is a *wavelet set*.

The last 2 are examples of wavelet set wavelets. Dai and Larson, and Hernàndez, Wang, and Weiss initiated the theory of wavelet set wavelets.

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Definition of a frame

Definition (Duffin and Schaeffer 1952)

$\{e_j\}_{j\in J} \subseteq H$, a Hilbert space, is a *frame* for H if there exist $0 < A \leq B < \infty$ such that

$$\forall f \in H, \qquad A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2. \tag{1}$$

- The maximal A and minimal B are the *optimal frame bounds*, The term *frame bound* will mean optimal frame bound.
- A frame is *tight* if $A = B$, and
- A frame is *Parseval* if $A = B = 1$.
- If the second inequality of (1) is true, but possibly not the first, then $\{e_j\}_{j\in J}$ is a *Bessel sequence*. In this case, we shall still refer to B as the upper frame bound.

Frame wavelets

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If $L \subseteq \widehat{\mathbb{R}}^d$ is measurable and $\mathcal{W}(\check{\mathbb{1}}_L)$ is a frame (resp., tight frame or Parseval frame) for $L^2(\mathbb{R}^d)$, then L is a *frame* (resp., *tight frame* or *Parseval frame*) *wavelet set*.

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(Frame) wavelet set characterization

Proposition

Let $K \subseteq \widehat{\mathbb{R}}^d$ be measurable. The following are equivalent:

- *K is a (Parseval frame) wavelet set.*
- *K is $\mathbb{Z}^d$-translation congruent to (a subset of) $[0,1)^d$, and K is dyadic-dilation congruent to $[-1,1)^d \backslash [-\frac{1}{2},\frac{1}{2})^d$.*
- *$\{K + k : k \in \mathbb{Z}^d\}$ is a partition of (a subset of) $\widehat{\mathbb{R}}^d$ and $\{2^n K : n \in \mathbb{Z}\}$ is a partition of $\widehat{\mathbb{R}}^d$.*

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Set up and construction

[Benedetto and Leon 1999 – Benedetto and Sumetkijakan 2006]

- $K_0 \subseteq \widehat{\mathbb{R}}^d$, bounded neighborhood of 0.
- K_0 is $\mathbb{Z}^d$-translation congruent to $[0,1]^d$.
- $S : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ injective, measurable such that

$$\forall \gamma \in \mathbb{R}^d, \exists k_\gamma \in \mathbb{Z}^d \quad \text{such that } S(\gamma) = \gamma + k_\gamma .$$

Definition

For each $m \in \mathbb{N} \cup \{0\}$ define

$$\begin{aligned} A_m &= K_m \cap \left[\bigcup_{n=1}^{\infty} 2^{-n} K_m\right], \\ K_{m+1} &= (K_m \backslash A_m) \cup S(A_m), \\ \text{and} \quad K &= \left[K_0 \backslash \bigcup_{m=0}^{\infty} A_m\right] \cup \left[\bigcup_{m=0}^{\infty} \left(S(A_m) \backslash \bigcup_{n>m} A_n\right)\right]. \end{aligned}$$

This process is the *neighborhood mapping construction*. Loosely speaking, K is the limit of the K_m.

Theorem and remark

Theorem (Benedetto with Leon and Sumetkijakan, 1998 – 2006)

a. Let K be defined by the neighborhood mapping construction. K is a wavelet set.
b. For each $m \geq 0$, $K_m \backslash A_m$ is a Parseval frame wavelet set, and, for each $m > 0$, K_m is a frame wavelet set with frame bounds 1 and 2.

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Daubechies-Christensen bounds

Theorem (Daubechies 92, Christensen 02)

Let $\psi \in L^2(\widehat{\mathbb{R}}^d)$, *and let* $a > 0$ *be arbitrary. Define*

$$\mu_\psi(\gamma) = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(2^n\gamma)\,\hat{\psi}(2^n\gamma+k)\right| \quad and$$

$$M_\psi = \operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d} \mu_\psi(\gamma) = \operatorname{esssup}_{a\le\|\gamma\|\le 2a} \mu_\psi(\gamma).$$

If $M_\psi < \infty$, *then* $\mathcal{W}(\psi)$ *is a Bessel sequence with upper frame bound* B, *and* $M_\psi \geq B$. *Similarly, define*

$$\nu_\psi(\gamma) = \left[\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(2^n\gamma)\right|^2 - \sum_{k\neq 0}\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(2^n\gamma)\,\hat{\psi}(2^n\gamma+k)\right|\right] \quad and$$

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Chui-Shi-Jing bounds

Proposition (Chui and Shi 1993, Jing 99)

Define $\kappa_\psi(\gamma) = \sum_{n\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\gamma\right)\right|^2$. *If* $\mathcal{W}(\psi)$ *is a wavelet frame for* $L^2(\mathbb{R}^d)$ *with bounds* A *and* B*, then, for almost all* $\gamma \in \widehat{\mathbb{R}}^d$*,*

$$A \leq \kappa_\psi(\gamma) \leq B.$$

Corollary

Let $\psi \in L^2(\mathbb{R}^d)$*. Let* $a > 0$ *be arbitrary. If* $M_\psi < \infty$*, then* $\mathcal{W}(\psi)$ *is a Bessel sequence with bound* B *satisfying*

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Definition

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An *approximate identity* is a family $\{k_{(\lambda)} : \lambda > 0\} \subseteq L^1(\mathbb{R}^d)$ of functions with the following properties:

i. $\forall \lambda > 0$, $\int k_{(\lambda)}(x)dx = 1$;

ii. $\exists K$ such that $\forall \lambda > 0$, $\|k_{(\lambda)}\|_{L^1(\mathbb{R}^d)} \leq K$;

iii. $\forall \eta > 0$, $\lim_{\lambda \to \infty} \int_{\|x\| \geq \eta} |k_{(\lambda)}(x)|dx = 0$.

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Well-known facts

Proposition

Suppose $k \in L^1(\mathbb{R}^d)$ *satisfies* $\int k(x)dx = 1$. *Define the family,*

$$\{k_\lambda : k_\lambda(x) = \lambda^d k(\lambda x), \lambda > 0\},$$

of dilations. Then, the following assertions hold.

a. $\{k_\lambda\}$ *is an approximate identity;*

b. *If* $f \in L^p(\mathbb{R}^d)$ *for some* $1 \leq p < \infty$, *then* $\lim_{\lambda\to\infty} \|f * k_\lambda - f\|_{L^p(\mathbb{R}^d)} = 0$;

c. *If* k *is an even function, there exists a subsequence* $\{\lambda_m\}$ *of* $\{\lambda\}$ *such that*

$$\lim_{m\to\infty} \int f(u) T_x k_{\lambda_m}(u)du = f(x) \;\; a.e. \;\; x \in \mathbb{R}^d.$$

Proof

Proof of *a*.

To verify the condition of Definition 13.*a*, we compute

$$\int k_\lambda(x)dx = \lambda^d \int k(\lambda x)dx = \int k(u)du = 1.$$

For part *b* we compute

$$\int |k_\lambda(x)|dx = \lambda^d \int |k(\lambda x)|dx = \int |k(u)|du = K < \infty,$$

where K is finite since $k \in L^1(\mathbb{R}^d)$. For part *c*, take $\eta > 0$ and compute

$$\int_{\|x\| \geq \eta} |k_\lambda(x)|dx = \lambda^d \int_{\|x\| \geq \eta} |k(\lambda x)|dx = \int_{\|u\| \geq \lambda\eta} |k(u)|du;$$

this last term tends to 0 as λ tends to ∞ since $\eta > 0$ and because of the definition of the integral.

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Proof cont.

Proof of *b*.

Setting $w = \lambda u$, we have

$$\begin{aligned} f * k_\lambda(x) - f(x) &= \int [f(x-u) - f(x)]\, k_\lambda(u)du \\ &= \int \left[f(x - \frac{w}{\lambda}) - f(x)\right] k(w)dw \\ &= \int \left[T_{\frac{w}{\lambda}} f(x) - f(x)\right] k(w)dw. \end{aligned}$$

Apply Minkowsi's inequality for integrals:

$$\|f * k_\lambda - f\|_p \leq \int \|T_{\frac{w}{\lambda}} f - f\|_p |k(w)|dw.$$

As $\|T_{\frac{w}{\lambda}} f - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $\lambda \to \infty$ for each w, the assertion follows from the dominated convergence theorem. □

Proof cont.

Proof of c.

The last part follows from the evenness of k.

$$\begin{aligned} \int f(u) T_x k_{\lambda_m}(u)du &= \int f(u) k_{\lambda_m}(u-x)du \\ &= \int f(u) k_{\lambda_m}(x-u)du \\ &= f * k_{\lambda_m}(x). \end{aligned}$$

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Canonical $K_0 \backslash A_0$

Example

Let $L = [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2})$, which is a $K_0 \backslash A_0$ Parseval frame wavelet set. Let $m \geq 5$ and define $\phi_m = \frac{m}{2} \mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]}$.

$\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is a wavelet frame with bounds

$$\frac{2}{9} \leq A \leq \frac{1}{4} \text{ and } B = \frac{17}{16}.$$

$\mathcal{W}(\mathbb{1}_L^\vee)$ is a Parseval frame and $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ in L^2, BUT the bounds do not change as $m \to \infty$.

The frame bounds are computed using the Chui-Shi-Jing bounds, Daubechies-Christensen bounds and another method using approximate identities introduced in [Benedetto K 08].

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Let $L = [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2})$, which is a $K_0 \backslash A_0$ Parseval frame wavelet set.
Let $m \geq 5$ and define $\phi_m = \frac{m}{2} \mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]}$.
$\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is a wavelet frame with bounds

$$\frac{2}{9} \leq A \leq \frac{1}{4} \text{ and } B = \frac{17}{16}.$$

$\mathcal{W}(\mathbb{1}_L^\vee)$ is a Parseval frame and $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ in L^2, BUT the bounds do not change as $m \to \infty$.

The frame bounds are computed using the Chui-Shi-Jing bounds, Daubechies-Christensen bounds and another method using approximate identities introduced in [Benedetto K 08].

Spline smoothing

Example

Let $L = [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2})$, as above.
Let $m \geq 13$ and consider the approximate identity $\{\phi_m : m > 12\}$, where

$$\phi_m(\gamma) = \max(m(1 - m|\gamma|), 0), \quad \gamma \in \widehat{\mathbb{R}}.$$

Then $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is a wavelet frame with bounds

$$\frac{2}{9} \leq A \leq \frac{1}{4} \text{ and } B = \frac{65}{64}.$$

Convolving with continuous splines gives upper frame bounds closer to 1, but the bounds are still uniformly bounded away from 1. We shall say there are *frame bound gaps*. (There is no Gibbs phenomenon using the Fejér kernel in Fourier series.)

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Questions to answer

- Do we obtain a frame when we try to smooth $K_1 \backslash A_1$ from the 1-d Journé neighborhood mapping construction?
- Can we ever precisely determine the lower frame bound?
- What happens when we smooth $K_0 \backslash A_0$ from higher dimensional Journé constructions?
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When smoothing fails

Example

Let

$$L = [-\frac{9}{4}, -2) \cup [-\frac{1}{2}, -\frac{9}{32}) \cup [\frac{9}{32}, \frac{1}{2}) \cup [2, \frac{9}{4}),$$

which is a $K_1 \backslash A_1$ Parseval frame wavelet set from the neighborhood mapping construction of the 1-d Journé set. For $m \in \mathbb{N}$, define

- Define $\phi_m = \frac{m}{2} \mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]}$.
- $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is not a frame for any m.
- $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is a Bessel sequence, and, for any $m \geq 65$, the Bessel bound is bounded between $\frac{305}{256}$ and $\frac{11}{8}$.

The fact that $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ is not a frame can be shown by considering a specific sequence of functions constructed from approximate identities in L^2; this is the same method we developed to obtain the bounds on A in the previous 2 examples.

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Frequency shrinking 1

Definition

For any measurable subset $L \subseteq \mathbb{R}^d$ define

$$\Delta(L) = \operatorname{dist}\Big(L, \bigcup_{k \in \mathbb{Z}^d \setminus \{0\}} (L+k)\Big).$$

Theorem (Benedetto and K 08, Chui and Shi 92)

Let $L \subseteq \widehat{\mathbb{R}}^d$ *be a bounded measurable set such that* $\Delta(L) > 0$, $\operatorname{dist}(0, L) > 0$, *and* $\bigcup_{n \in \mathbb{Z}} 2^n L = \widehat{\mathbb{R}}^d$ *up to a set of measure* 0. *If* $\hat{\psi} \in L^\infty(\widehat{\mathbb{R}}^d)$ *with*

$$\mathbf{supp}\,\hat{\psi} = \{\gamma \in \widehat{\mathbb{R}}^d : \hat{\psi}(\gamma) \neq 0\} = L,$$

then $\mathcal{W}(\psi)$ *is a frame for* $L^2(\mathbb{R}^d)$. *The frame bounds are* $\operatorname{essinf}_\gamma \kappa_\psi(\gamma)$ *and* $\operatorname{esssup}_\gamma \kappa_\psi(\gamma)$.

Frequency shrinking 2

If the L is measurable and the closure $\overline{L} \subseteq (-\frac{1}{2}, \frac{1}{2})^d$, then L is bounded and $\Delta(L) > 0$.

Corollary (Benedetto and K 08)

Let L be a Parseval frame wavelet set from the neighborhood mapping construction. Let $\delta = \mathrm{dist}(0, L) > 0$. Let $\alpha > 0$ be such that the closure $\alpha\overline{L} \subseteq (-\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)^d$, for some $0 < \epsilon < \frac{1}{2}$. Further let ϕ be an essentially bounded non-negative function whose support lies in $\min\{\frac{\alpha\delta}{2}, \epsilon\} \cdot (-1, 1)^d$ and contains a neighborhood about the origin. Then if $\hat{\psi} = \mathbb{1}_{\alpha L} * \phi$, $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$.

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Frequency shrinking example 1

Example

Let

$$L = \left[-\frac{9}{32}, -\frac{1}{4}\right) \cup \left[-\frac{1}{16}, -\frac{9}{256}\right) \cup \left[\frac{9}{256}, \frac{1}{16}\right) \cup \left[\frac{1}{4}, \frac{9}{32}\right).$$

Then L is $K_1 \backslash A_1$ from the 1-d Journé construction, shrunk by a factor of 8. Further let $\hat{\psi}_m = \mathbb{1}_L * \frac{n}{2} \mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]}$. Then for any $m \geq 384$, $\mathcal{W}(\psi_m)$ is frame with bounds $\frac{81}{260}$ and $\frac{305}{256}$. Note that $\mathcal{W}((\mathbb{1}_{8L} * \frac{m}{2} \mathbb{1}_{[-\frac{1}{m}, \frac{1}{m}]})^\vee)$ is not a frame for any $m > 0$.

Frequency shrinking example 2

Example

Let $L_a = \left[-a, -\frac{a}{2}\right) \cup \left[\frac{a}{2}, a\right)$ for $0 < a < \frac{1}{2}$. Then L_a is $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)$ from the 1-d Journé construction, dilated by a factor of $2a < 1$. Recall that

$$\mathcal{W}((\mathbb{1}_{[-\frac{1}{2},-\frac{1}{4})\cup[\frac{1}{4},\frac{1}{2})} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m})})^{\vee})$$

is a frame with upper frame bound $\frac{17}{16}$ and lower frame bound between $\frac{2}{9}$ and $\frac{1}{4}$. Define $\hat{\psi}_{m,a} = \mathbb{1}_{L_a} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. For $0 < a < \frac{1}{2}$ and $m \geq \max\{\frac{2}{1-2a}, \frac{6}{a}\}$, $\mathcal{W}(\psi_{m,a})$ is a frame with with frame bounds $\frac{9}{20}$ and $\frac{17}{16}$.

A heuristic argument

For $0 < \alpha$ and $\psi \in L^2(\mathbb{R}^d)$, let $\hat{\varphi}(\gamma) = \hat{\psi}(\alpha\gamma)$. Then $\mathcal{F}\varphi = \alpha^{-d/2} D_{\log_2 \alpha} \mathcal{F}\psi$,

$$\begin{aligned} \Rightarrow \varphi &= \mathcal{F}^{-1}\mathcal{F}\varphi \\ &= \mathcal{F}^{-1}(\alpha^{-d/2} D_{\log_2 \alpha}\mathcal{F})\psi \\ &= \mathcal{F}^{-1}(\alpha^{-d/2}\mathcal{F} D_{-\log_2 \alpha})\psi \\ &= \alpha^{-d/2} D_{-\log_2 \alpha}\psi \\ \Rightarrow D_n T_k \varphi &= \alpha^{-d/2} D_n T_k D_{-\log_2 \alpha}\psi \\ &= \alpha^{-d/2} D_{n-\log_2 \alpha} T_{\frac{k}{\alpha}}\psi. \end{aligned}$$

Hence if $\alpha = 2^N$, for $N \in \mathbb{N}$,

$$\text{span}\{D_n T_k \varphi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} = \text{span}\{D_n T_{\frac{k}{2^N}}\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

One may call this an *oversampling* of the continuous wavelet system $\{D_{\log_2 r} T_s \psi : r > 0, s \in \mathbb{R}\}$.

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Oversampling

Theorem (Chui and Shi 92)

Let $\mathcal{W}(\psi)$ be a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Then for every odd positive integer N, the family

$$\{D_n T_{\frac{k}{N}} \psi : n, k \in \mathbb{Z}\}$$

is a frame with bounds $\tilde{A}$ and $\tilde{B}$ which satisfy $\tilde{A} \geq NA$ and $\tilde{B} \leq NB$.

Theorem (Bendetto and K 08)

There exist wavelet frames $\mathcal{W}(\psi)$ for $L^2(\mathbb{R}^d)$ with bounds A and B such that

- $\{D_n T_{\frac{k}{2^N}} \psi(x) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ *is a frame with lower bound $\tilde{A} \neq A$.*
- $\{D_n T_{\frac{k}{2^N}} \psi(x) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ *has the same frame bounds as $\mathcal{W}(\psi)$.*

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Oversampling cont.

Theorem (Chui and Shi 92)

Let $\psi \in L^2(\mathbb{R})$ decay sufficiently fast and satisfy $\int \psi(x)dx = 0$. If $\mathcal{W}(\psi)$ forms a frame, then for any positive integer N,

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is a frame also.

Theorem (Bendetto and K 08)

There exist wavelet systems $\mathcal{W}(\psi)$ in $L^2(\mathbb{R}^d)$ which are not frames, but are such that $\{D_n T_{\frac{k}{\alpha}} \psi(x) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ is a frame for some $\alpha > 0$.

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Frame bound gap

Definition

Let $\psi \in L^2(\widehat{\mathbb{R}}^d)$ be a Parseval frame wavelet and $\{\psi_m\}_{m\in\mathbb{N}} \subseteq L^2(\widehat{\mathbb{R}}^d)$ be a sequence of frame wavelets (or wavelet systems which are Bessel sequences) with lower frame bounds A_m and upper frame bounds B_m (or just upper frame bounds B_m) for which

$$\lim_{m\to\infty} \|\psi - \psi_m\|_{L^2(\widehat{\mathbb{R}}^d)} = 0.$$

If $\overline{\lim}_{m\to\infty} A_m < 1$ or $\underline{\lim}_{m\to\infty} B_m > 1$, then there is a *frame bound gap*. By Parseval's equality, $\|\psi - \psi_m\|_{L^2(\widehat{\mathbb{R}}^d)} = \|\hat{\psi} - \hat{\psi}_m\|_{L^2(\widehat{\mathbb{R}}^d)}$, so it suffices to check for convergence on the frequency domain.

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A remark

Let $L \subseteq \widehat{\mathbb{R}}^d$ be bounded and measurable and $g \in L^1_{loc}(\widehat{\mathbb{R}}^d)$. For $m > 1$ define

$$\begin{aligned} g_{(m)}(\gamma) &= mg(m\gamma), \text{ and} \\ \hat{\psi}_m &= \mathbb{1}_L * g_{(m)}. \end{aligned}$$

Then

$$\begin{aligned} \hat{\psi}_m(u) &= \int \mathbb{1}_L(u-\gamma)g_{(m)}(\gamma)d\gamma \\ &= \int \mathbb{1}_L(u-\frac{\gamma}{m})g(\gamma)d\gamma \\ &= \int_{-mL+mu} g(\gamma)d\gamma. \end{aligned}$$

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Boxy-torus result

Theorem (Benedetto and K 08)

For $0 < a < 1/2$, *let* $L \subseteq \widehat{\mathbb{R}}^d$ *be the Parseval frame wavelet set* $[-a,a]^d \backslash [-\frac{a}{2}, \frac{a}{2}]^d$. *Also let* $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ *satisfy the following conditions:*

- *i.* $\operatorname{supp} g \subseteq \prod_{i=1}^d [-b_i, c_i]$, *where for all* i, $b_i, c_i > 0$;
- *ii.* $\int g(\gamma) d\gamma = 1$; *and*
- *iii.* $0 < \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma < 1$ *and* $0 < \int_{\prod_{i=1}^d [-\frac{b_i}{2}, c_i]} g(\gamma) d\gamma < 1$.

Define $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$. *For any*

$$m > \max_{1 \le i \le d} \{\max \left\{ \frac{2(b_i + c_i)}{a}, \frac{b_i + c_i}{1 - 2a}, \frac{4b_i + c_i}{a}, \frac{4c_i + b_i}{a} \right\}\},$$

$\mathcal{W}(\psi_m)$ *is a frame with frame bounds* A_m *and* B_m, *and there exist* $\alpha < 1$ *and* $\beta > 1$, *both independent of* m, *such that* $A_m \le \alpha$ *and* $B_m \ge \beta$. *In particular, there are frame bound gaps.*

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Boxy-torus result cont.

- Any non-negative function $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ which integrates to 1 and has support $\operatorname{supp} g = \prod_{i=1}^{d} [-b_i, c_i]$ satisfies the hypotheses.
- This result holds true if $m \in \mathbb{N}$ or $m \in \mathbb{R}$.

Sketch.

Since m is large enough, we may apply some of the preceding theorems to conclude that $\mathcal{W}(\psi_m)$ is a frame.
We then estimate the Chui-Shi bounds. □

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Boxy-torus result cont.

- Any non-negative function $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ which integrates to 1 and has support $\operatorname{supp} g = \prod_{i=1}^{d} [-b_i, c_i]$ satisfies the hypotheses.
- This result holds true if $m \in \mathbb{N}$ or $m \in \mathbb{R}$.

Sketch.

Since m is large enough, we may apply some of the preceding theorems to conclude that $\mathcal{W}(\psi_m)$ is a frame.
We then estimate the Chui-Shi bounds. □

Frame bound convergence

Corollary

For $0 < a < \frac{1}{2}$*, let* $L_d \subseteq \widehat{\mathbb{R}}^d$ *be the wavelet set* $[-a,a]^d \backslash [-\frac{a}{2}, \frac{a}{2}]^d$*. Also, let* $g : \widehat{\mathbb{R}} \to \mathbb{R}$ *satisfy the following conditions:*

- i. $\operatorname{supp} g \subseteq [-b, c]$ *for some* $b, c > 0$*;*
- ii. $\int g(\gamma) d\gamma = 1$*; and*
- iii. $0 < \int_{\frac{c}{2}}^{c} g(\gamma) d\gamma < 1$ *and* $0 < \int_{-\frac{b}{2}}^{c} g(\gamma) d\gamma < 1$*.*

Define $g_d = \bigotimes_{i=1}^{d} g : \widehat{\mathbb{R}}^d \to \mathbb{R}$*. Further define* $\hat{\psi}_{m,d} = \mathbb{1}_{L_d} * g_{d_{(m)}}$*. Then for large enough* m *and* $d \geq 1$*,* $\mathcal{W}(\psi_{m,d})$ *is a frame with bounds* $A_{m,d}$ *and* $B_{m,d}$ *which are bounded away from* 1 *uniformly over* m*. For such* m*,* $\lim_{d\to\infty} B_{m,d} = 2$*.*

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Frame bound convergence, cont.

Proof.

Using the same methods as above, we may determine that

$$\begin{aligned} A_{m,d} &\le \left(1-\left(\int_{\frac{c}{2}}^{c} g(\gamma)d\gamma\right)^{d}\right)^{2} < 1, \text{ and} \\ B_{m,d} &\ge \left(1-\left(\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma\right)^{d}\right)^{2} + 1 > 1. \end{aligned}$$

□

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Frame bound convergence, cont.

Proof.

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□

Frame bound gaps for $\widehat{\mathbb{R}}$ wavelet sets

Theorem (Benedetto and K 08)

Let $L = \bigcup_{u \in \mathcal{J} \subseteq \mathbb{Z}} [a_j, b_j]$ *be a Parseval frame wavelet set. Let* $g : \widehat{\mathbb{R}} \to \mathbb{R}$ *be a non-negative function satisfying* $\int g(\gamma) d\gamma = 1$ *and with support* $\operatorname{supp} g = [-c, d]$*, where* $c, d > 0$*. Define* $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$ *for* $m > \frac{c+d}{b_j - a_j}$ *for all* $j \in \mathcal{J}$*. Then if* $\mathcal{W}(\psi_m)$ *forms a Bessel sequence, the upper frame bound satisfies* $B_m \geq \beta > 1$*, where* β *is independent of* m*. In particular, there is a frame bound gap.*

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$\mathcal{S}$ Parseval frames

Theorem (Han, early 90s)

Suppose that a Parseval frame wavelet set $L \in \widehat{\mathbb{R}}$ *is such that*

$$\operatorname{dist}(L, \cup_{k\in\mathbb{Z}\setminus\{0\}}(L+k)) > 0$$

Let $m > 0$*. There exists* $\psi_m \in \mathcal{S}(\mathbb{R})$ *such that* $\hat{\psi}_m \in C_c^\infty(\mathbb{R})$*,* $\mathcal{W}(\psi)$ *is a Parseval frame, and* $\hat{\mathbb{1}}_L = \hat{\psi}_m$ *except on a set of measure* $\frac{1}{m}$*.*

Corollary (K 08)

Let L *be a set* $K_n \setminus A_n$ *from a* 1*-*d *implementation of the neighborhood mapping construction. Let* $\delta > 0$ *and* $m \in \mathbb{Z}$ *be such that* $\operatorname{clos}\{2^m L\} \subseteq (-\frac{1}{2}+\delta, \frac{1}{2}-\delta)^d$*. Then there exists* $\psi \in \mathcal{S}(\mathbb{R})$ *such that* $\hat{\psi} \in C_c^\infty(\mathbb{R})$ *and* $\mathcal{W}(\psi)$ *is a Parseval frame and the measure of* $\operatorname{supp}(\psi)\setminus L$ *is arbitrarily small.*

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C^∞ partitions of unity

- In topology and analysis, *partitions of unity*, especially those which are smooth are particularly useful.
- Let $f(\gamma) = e^{-1/\gamma}\mathbb{1}_{[0,\infty)}(\gamma)$ and $0 < a < b$. A common *bump function* employed in such a construction is

$$\frac{f(b-|\gamma|)}{f(b-|\gamma|)+f(|\gamma|-a)}.$$

- We can use these functions to define smooth frames which have frame bounds which converge to 1.

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